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Generalized Cowin–Mehrabadi theorems and a direct proof that the number of linear elastic symmetries is eight

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This paper is dedicated to Prof. George Dvorak

Abstract

The Cowin–Mehrabadi theorem is generalized to allow less restrictive and more flexible conditions for locating a symmetry plane in an anisotropic elastic material. The generalized theorems are then employed to prove that the number of linear elastic symmetries is eight. The proof starts by imposing a symmetry plane to a triclinic material and, after new elastic symmetries are found, another symmetry plane is imposed. This process exhausts all possibility of elastic symmetries, and shows that there are only eight elastic symmetries. At each stage when a new symmetry plane is added, explicit results are obtained for the locations of the new symmetry plane that lead to a new elastic symmetry. It takes as few as three, and at most five, symmetry planes to reduce a triclinic material (which has no symmetry plane) to an isotropic material for which any plane is a symmetry plane.

Keywords: Anisotropic materials; Material symmetry; Elastic symmetries; Cubic materials

1. Introduction

In a linear anisotropic elastic material, let **n** be a unit vector normal to the plane of material symmetry. If **m** is any vector on the symmetry plane, a set of necessary and sufficient conditions for **n** to be normal to the symmetry plane is (Cowin and Mehrabadi, 1987)

$$C_{ijkk}n_j = (C_{pqtt}n_pn_q)n_i, \tag{1.1}$$

$$C_{isks}n_k = (C_{pqrq}n_pn_r)n_i, \tag{1.2}$$

$$C_{ijks}n_jn_sn_k = (C_{pqrt}n_pn_qn_rn_t)n_i, \tag{1.3}$$

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$$C_{ijks}m_j m_s n_k = (C_{pqrt}n_p m_q n_r m_t)n_i, \tag{1.4}$$

in which repeated indices imply summation and C_{ijks} is the elastic stiffness which is assumed to possess the full symmetry

$$C_{ijks} = C_{jiks} = C_{ijsk} = C_{ksij}. \tag{1.5}$$

If we define the 3×3 matrices **U**, **V**, **Q**(**n**) whose elements are given by

$$U_{ij} = C_{ijkk}, \quad V_{ik} = C_{isks}, \quad Q_{ik}(\mathbf{n}) = C_{ijks}n_j n_s, \tag{1.6}$$

(1.1)–(1.4) say that **n** is an eigenvector of **U**, **V**, **Q**(**n**) and **Q**(**m**) for any **m** on the symmetry plane. In a separate paper, Cowin and Mehrabadi (1989) (see also Norris, 1989) showed that (1.3) and (1.4) are necessary and sufficient conditions. Other alternate necessary and sufficient conditions can be found in Ting (1996, pp. 60–62).

In Section 2, the Cowin–Mehrabadi theorem is generalized. It is shown that there is no need for **n** to satisfy (1.4) for *all* **m**. It suffices to satisfy (1.4) for any *one* **m** when **n** is an eigenvector of **U**, **V** and **Q**(**n**). If **n** is an eigenvector of two of the three matrices **U**, **V**, **Q**(**n**), it suffices to satisfy (1.4) for any *two* distinct **m** (with one exception that the two **m** be nonorthogonal as shown in Theorem IIc). When **n** is an eigenvector of any one of **U**, **V**, **Q**(**n**), it suffices to satisfy (1.4) for any *three* distinct **m**. In particular, (1.3) and (1.4) with any three distinct **m** are necessary and sufficient conditions for **n** to be normal to a symmetry plane. This feature is useful because, for any **n**, one can always choose three **m** that are on the coordinate planes. Thus at least one of the three components of **m** vanishes. This simplifies the computation. The vectors **n** and **m** need not be unit vectors because the scales of **n** and **m** can be absorbed in the eigenvalue.

It is known (Voigt, 1910; Nye, 1957) that there are eight elastic symmetries for a linear anisotropic elastic material. Ting (1996, Chapter 2) has shown how to reduce all eight elastic symmetries by imposing as few as three symmetry planes. He did not investigate if there are other elastic symmetries. Forte and Vianello (1996) and Chadwick et al. (2001) proved that the number of elastic symmetries is eight. In this paper we employ a more direct approach using the generalized Cowin–Mehrabadi theorems presented in Section 2. The derivation is an expansion of the one employed in Ting (1996, Chapter 2). It resembles the one in Chadwick et al. (2001) when the first and the second symmetry planes are added. The derivations are different when the third symmetry plane is added.

The structure of the elastic stiffness matrix for a monoclinic material that has one symmetry plane is presented in Section 2. In Section 3 a second symmetry plane is added to the monoclinic material. Depending on the location of the symmetry plane one obtains orthotropic, tetragonal, trigonal or transversely isotropic material. In Sections 4–7 a third symmetry plane is added to the ones presented in Section 3. Again, depending on the location of the third symmetry plane it may produce one of the four elastic symmetries obtained in Section 3 or a cubic or isotropic material. A transversely isotropic or a cubic material reduces to an isotropic material when any symmetry plane is added. Thus there are a total of eight elastic symmetries, namely, triclinic, monoclinic, orthotropic, tetragonal, trigonal, transversely isotropic, cubic and isotropic. The novel feature of the proof is that the plane added at each step is completely arbitrary. Thus all possible choices of symmetry planes are exhausted at each step. An interesting result obtained is the structure of the elastic matrix of a cubic material. It can resemble a tetragonal or a trigonal material (Chadwick et al., 2001).

2. Generalized Cowin–Mehrabadi theorems

To present various sets of necessary and sufficient conditions for **n** to be normal to a symmetry plane, we assume without loss in generality that $x_1 = 0$ is the plane of symmetry. Hence let

$$n_i = \delta_{i1}, \quad m_i = \delta_{i2}\cos\theta + \delta_{i3}\sin\theta, \tag{2.1}$$

where θ is an arbitrary constant and δ_{ij} is the Kronecker delta. Eqs. (1.1)–(1.4) are trivial identities when $i = 1$. For $i \neq 1$ they reduce, respectively, to

$$\begin{aligned} &C_{i1kk} = 0, \quad C_{is1s} = 0, \quad C_{i111} = 0, \\ &C_{i212}\cos^2\theta + (C_{i213} + C_{i312})\cos\theta\sin\theta + C_{i313}\sin^2\theta = 0. \end{aligned} \tag{2.2}$$

In the contracted notation $C_{\alpha\beta}$, (2.2) for $i = 2, 3$ are

$$C_{16} + C_{26} + C_{36} = C_{15} + C_{25} + C_{35} = 0, \tag{2.3}$$

$$C_{16} + C_{26} + C_{45} = C_{15} + C_{46} + C_{35} = 0, \tag{2.4}$$

$$C_{16} = C_{15} = 0, \tag{2.5}$$

and

$$\begin{aligned} &C_{26}\cos^2\theta + (C_{25} + C_{46})\cos\theta\sin\theta + C_{45}\sin^2\theta = 0, \\ &C_{46}\cos^2\theta + (C_{45} + C_{36})\cos\theta\sin\theta + C_{35}\sin^2\theta = 0. \end{aligned} \tag{2.6}$$

Thus (2.3)–(2.6) replace (1.1)–(1.4) when $x_1 = 0$ is the plane of symmetry. It is known (Voigt, 1910; Love, 1927) that, for a *monoclinic* material with the symmetry plane at $x_1 = 0$,

$$C_{15} = C_{16} = C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0. \tag{2.7}$$

The structure of the 6×6 matrix $C_{\alpha\beta}$ that satisfies (2.7) is

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{2.8}$$

Only the upper triangle of the matrix is shown since $\mathbf{C}$ is symmetric. With (2.7), (2.3)–(2.6) are trivially satisfied for all θ. Thus $\mathbf{n}$ being an eigenvector of $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Q(n)}$, $\mathbf{Q(m)}$ for all $\mathbf{m}$ is a sufficient condition for $\mathbf{n}$ to be normal to the symmetry plane. However, it is not a necessary condition. We derive necessary conditions below.

When $\mathbf{n}$ is an eigenvector of $\mathbf{U}$, $\mathbf{Un}$ is proportional to $\mathbf{n}$. This means that $\mathbf{m}^T\mathbf{Un} = 0$ for any two $\mathbf{m}$, which is (2.3). Thus $\mathbf{n}$ being an eigenvector of a matrix imposes two conditions on $C_{\alpha\beta}$. When $\mathbf{n}$ is an eigenvector of $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Q(n)}$, $\mathbf{Q(m)}$, we have eight conditions on $C_{\alpha\beta}$ given in (2.3)–(2.6). We show that, of the four matrices we need for $\mathbf{n}$ to be an eigenvector, we can choose one, two or three matrices among $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Q(n)}$. The remaining matrices are supplied by $\mathbf{Q(m)}$ for different $\mathbf{m}$. They are discussed separately below.

When $\mathbf{n}$ is an eigenvector of $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Q(n)}$ and $\mathbf{Q(m)}$, (2.3)–(2.6) apply. With (2.5), (2.3) and (2.4) give

$$C_{26} + C_{36} = C_{25} + C_{35} = C_{26} + C_{45} = C_{46} + C_{35} = 0, \tag{2.9}$$

and (2.6) is

$$C_{26}\cos 2\theta - C_{35}\sin 2\theta = 0, \quad C_{26}\sin 2\theta + C_{35}\cos 2\theta = 0. \tag{2.10}$$

The two equations in (2.10) hold only when

$$C_{26} = C_{35} = 0. \tag{2.11}$$

Together with (2.9) we obtain (2.7). Eq. (2.10) leads to (2.11) for any θ. Hence **n** needs to be an eigenvector of **Q(m)** for only one **m**. This proves the following theorem.

Theorem I. *A necessary and sufficient condition for* **n** *to be normal to the symmetry plane is that* **n** *be an eigenvector of* **U**, **V**, **Q(n)** *and* **Q(m)** *for any one* **m**.

We next consider the case when **n** is an eigenvector of two of the three matrices **U**, **V** and **Q(n)**. We then need **n** an eigenvector of **Q(m)** for two **m**.

Let **n** be an eigenvector of **U**, **V** and **Q(m)** for any two **m**. In this case (2.3), (2.4) and (2.6) hold for $\theta = \theta_1, \theta_2$, say. When (2.3) and (2.4) hold, $C_{36} = C_{45}$ and $C_{25} = C_{46}$, and (2.6) can be written as

$$\begin{aligned} &C_{26}\cos^2\theta + C_{46}\sin 2\theta + C_{45}\sin^2\theta = 0,\\ &C_{46}\cos^2\theta + C_{45}\sin 2\theta + C_{35}\sin^2\theta = 0. \end{aligned} \tag{2.12}$$

Application of (2.12) for $\theta = \theta_1, \theta_2$ gives four equations that can be written as

$$\begin{bmatrix} \cos^2\theta_1 & \sin^2\theta_1 & \sin 2\theta_1 & 0 \\ \cos^2\theta_2 & \sin^2\theta_2 & \sin 2\theta_2 & 0 \\ 0 & \sin 2\theta_1 & \cos^2\theta_1 & \sin^2\theta_1 \\ 0 & \sin 2\theta_2 & \cos^2\theta_2 & \sin^2\theta_2 \end{bmatrix} \begin{bmatrix} C_{26} \\ C_{45} \\ C_{46} \\ C_{35} \end{bmatrix} = \mathbf{0}. \tag{2.13}$$

The determinant of the 4×4 matrix can be shown to be $\sin^4(\theta_1 - \theta_2)$, which is nonzero when the two **m** are distinct. The vectors **m** and $-$**m** are not considered distinct because if **n** is an eigenvector of **Q(m)**, it is an eigenvector of **Q**($-$**m**). Hence when the two **m** are distinct, (2.13) gives

$$C_{26} = C_{45} = C_{46} = C_{35} = 0. \tag{2.14}$$

Together with (2.3) and (2.4) we obtain (2.7). We therefore have the theorem:

Theorem IIa. *A necessary and sufficient condition for* **n** *to be normal to the symmetry plane is that* **n** *be an eigenvector of* **U**, **V** *and* **Q**1(**m**) *for any two distinct* **m**.

Let **n** be an eigenvector of **U**, **Q(n)** and **Q(m)** for any two **m**. In this case (2.3), (2.5) and (2.6) hold for $\theta = \theta_1, \theta_2$. When (2.3) and (2.5) hold, $C_{36} = -C_{26}$ and $C_{25} = -C_{35}$. Eq. (2.6) gives four equations for $\theta = \theta_1, \theta_2$, which can be written as

$$\begin{bmatrix} \cos^2\theta_1 & \sin^2\theta_1 & \frac{1}{2}\sin 2\theta_1 & \frac{-1}{2}\sin 2\theta_1 \\ \cos^2\theta_2 & \sin^2\theta_2 & \frac{1}{2}\sin 2\theta_2 & \frac{-1}{2}\sin 2\theta_2 \\ \frac{-1}{2}\sin 2\theta_1 & \frac{1}{2}\sin 2\theta_1 & \cos^2\theta_1 & \sin^2\theta_1 \\ \frac{-1}{2}\sin 2\theta_2 & \frac{1}{2}\sin 2\theta_2 & \cos^2\theta_2 & \sin^2\theta_2 \end{bmatrix} \begin{bmatrix} C_{26} \\ C_{45} \\ C_{46} \\ C_{35} \end{bmatrix} = \mathbf{0}. \tag{2.15}$$

The determinant of the 4×4 matrix can be shown to be $\sin^2(\theta_1 - \theta_2)$, which is nonzero when the two **m** are distinct. Hence when the two **m** are distinct, (2.15) leads to (2.14). Together with (2.3) and (2.5) we obtain (2.7).

Theorem IIb. *A necessary and sufficient condition for* **n** *to be normal to the symmetry plane is that* **n** *be an eigenvector of* **U**, **Q(n)** *and* **Q(m)** *for any two distinct* **m**.

Let **n** be an eigenvector of **V**, **Q(n)** and **Q(m)** for any two **m**. In this case (2.4)–(2.6) hold for $\theta = \theta_1, \theta_2$. When (2.4) and (2.5) hold, $C_{45} = -C_{26}$ and $C_{46} = -C_{35}$. Eq. (2.6) gives four equations for $\theta = \theta_1, \theta_2$, which can be written as

$$\begin{bmatrix} \cos 2\theta_1 & \frac{1}{2}\sin 2\theta_1 \\ \cos 2\theta_2 & \frac{1}{2}\sin 2\theta_2 \end{bmatrix} \begin{bmatrix} C_{26} \\ C_{25} - C_{35} \end{bmatrix} = \mathbf{0}, \quad \begin{bmatrix} \cos 2\theta_1 & \frac{1}{2}\sin 2\theta_1 \\ \cos 2\theta_2 & \frac{1}{2}\sin 2\theta_2 \end{bmatrix} \begin{bmatrix} C_{35} \\ C_{26} - C_{36} \end{bmatrix} = \mathbf{0}. \tag{2.16}$$

The determinant of the 2×2 matrix is $\frac{1}{2}\sin[2(\theta_1 - \theta_2)]$, which is nonzero when the two $\mathbf{m}$ are not orthogonal. Hence when the two $\mathbf{m}$ are not orthogonal,

$$C_{26} = C_{35} = C_{36} = C_{25} = 0. \tag{2.17}$$

Together with (2.4) and (2.5) we obtain (2.7).

Theorem IIc. *A necessary and sufficient condition for* $\mathbf{n}$ *to be normal to the symmetry plane is that* $\mathbf{n}$ *be an eigenvector of* $\mathbf{V}$, $\mathbf{Q}(\mathbf{n})$ *and* $\mathbf{Q}(\mathbf{m})$ *for any two distinct nonorthogonal* $\mathbf{m}$.

Finally, we consider the case when $\mathbf{n}$ is an eigenvector of $\mathbf{U}$, $\mathbf{V}$ or $\mathbf{Q}(\mathbf{n})$. We then need $\mathbf{n}$ an eigenvector of $\mathbf{Q}(\mathbf{m})$ for three $\mathbf{m}$. When $\mathbf{n}$ is an eigenvector of $\mathbf{Q}(\mathbf{m})$ for three $\mathbf{m}$ associated with $\theta = \theta_1, \theta_2, \theta_3$, (2.6) can be written as

$$\mathbf{K}\begin{bmatrix} C_{26} \\ C_{25} + C_{46} \\ C_{45} \end{bmatrix} = \mathbf{0}, \quad \mathbf{K}\begin{bmatrix} C_{46} \\ C_{45} + C_{36} \\ C_{35} \end{bmatrix} = \mathbf{0}, \tag{2.18}$$

where

$$\mathbf{K} = \begin{bmatrix} \cos^2\theta_1 & \frac{1}{2}\sin 2\theta_1 & \sin^2\theta_1 \\ \cos^2\theta_2 & \frac{1}{2}\sin 2\theta_2 & \sin^2\theta_2 \\ \cos^2\theta_3 & \frac{1}{2}\sin 2\theta_3 & \sin^2\theta_3 \end{bmatrix}. \tag{2.19}$$

The determinant of $\mathbf{K}$ is

$$|\mathbf{K}| = \sin(\theta_1 - \theta_2)\sin(\theta_2 - \theta_3)\sin(\theta_3 - \theta_1). \tag{2.20}$$

It is nonzero when the three $\mathbf{m}$ are distinct. Eq. (2.18) then yields

$$C_{26} = C_{46} = C_{45} = C_{35} = C_{25} = C_{36} = 0. \tag{2.21}$$

Together with (2.3), (2.4) or (2.5), we obtain (2.7).

Theorem III. *A necessary and sufficient condition for* $\mathbf{n}$ *to be normal to the symmetry plane is that* $\mathbf{n}$ *be an eigenvector of* $\mathbf{U}$, $\mathbf{V}$ *or* $\mathbf{Q}(\mathbf{n})$, *and an eigenvector of* $\mathbf{Q}(\mathbf{m})$ *for any three distinct* $\mathbf{m}$.

Chadwick et al. (2001, p. 2476) proved a *test* which says that $\mathbf{n}$ is normal to a symmetry plane when it is an eigenvector of $\mathbf{Q}(\mathbf{n})$ and $\mathbf{Q}(\mathbf{m})$ for *all* $\mathbf{m}$. The test is a special case of Theorem III. According to Theorem III, $\mathbf{n}$ need to be an eigenvector of $\mathbf{Q}(\mathbf{n})$ and $\mathbf{Q}(\mathbf{m})$ for only *three* $\mathbf{m}$. This fact, however, does not affect the validity of their analysis.

The fact that $\mathbf{m}$ can be chosen arbitrarily is an attractive feature. The computation of $\mathbf{Q}(\mathbf{n})$ is cumbersome when all three components of $\mathbf{n}$ are nonzero. In contrast, the computation of $\mathbf{Q}(\mathbf{m})$ is simplified because $\mathbf{m}$ can be chosen to lie on a coordinate plane so that at least one component of $\mathbf{m}$ vanishes. There are three such $\mathbf{m}$ on a symmetry plane. A good choice of four matrices for $\mathbf{n}$ to be an eigenvector is $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{Q}(\mathbf{m})$ for two distinct $\mathbf{m}$ because $\mathbf{U}$ and $\mathbf{V}$ do not involve $\mathbf{n}$ and $\mathbf{m}$.

It is shown in Ting (1996, p. 64) that $\mathbf{U}$ and $\mathbf{V}$ share the same set of eigenvectors when $\mathbf{UV}$ is symmetric. The eigenvectors of a symmetric matrix are orthogonal to each other so that if two eigenvectors of $\mathbf{U}$ and $\mathbf{V}$ are identical, the third eigenvectors are also identical. Thus $\mathbf{UV}$ must be symmetric when the material has more than one symmetry plane. If **UV** is not symmetric, the material can have at most one symmetry plane. However, **UV** can be symmetric for materials with one or no symmetry plane.

A *triclinic* material has no symmetry plane. When the material has one symmetry plane we may take, without loss in generality, $x_1 = 0$ as the symmetry plane. The material is *monoclinic* with the symmetry plane at $x_1 = 0$ shown in (2.7) or (2.8).

3. Adding a second symmetry plane

We now add a second symmetry plane. Without loss in generality let the normal **n** to the second symmetry plane lie on the plane $x_3 = 0$. Thus, (Fig. 1)

$$\mathbf{n}^{\mathrm{T}} = [\cos\theta \quad \sin\theta \quad 0] \tag{3.1}$$

for any $\theta \neq 0$ because $\theta = 0$ represents the first symmetry plane. We employ Theorem IIa so that **n** is an eigenvector of **U**, **V** and **Q(m)** for two **m**. This means that

$$\begin{aligned} &\mathbf{m}_1^{\mathrm{T}}\mathbf{U}\mathbf{n} = 0, \quad \mathbf{m}_1^{\mathrm{T}}\mathbf{V}\mathbf{n} = 0, \quad \mathbf{m}_1^{\mathrm{T}}\mathbf{Q}(\mathbf{m}_1)\mathbf{n} = 0, \quad \mathbf{m}_1^{\mathrm{T}}\mathbf{Q}(\mathbf{m}_2)\mathbf{n} = 0, \\ &\mathbf{m}_2^{\mathrm{T}}\mathbf{U}\mathbf{n} = 0, \quad \mathbf{m}_2^{\mathrm{T}}\mathbf{V}\mathbf{n} = 0, \quad \mathbf{m}_2^{\mathrm{T}}\mathbf{Q}(\mathbf{m}_1)\mathbf{n} = 0, \quad \mathbf{m}_2^{\mathrm{T}}\mathbf{Q}(\mathbf{m}_2)\mathbf{n} = 0, \end{aligned} \tag{3.2}$$

where we choose

$$\mathbf{m}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{m}_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix}. \tag{3.3}$$

With the $C_{\alpha\beta}$ given in (2.8), the third, second, first and fourth equations of (3.2), in that order, give

$$C_{34} = 0, \quad C_{14} = -C_{24} = C_{56} \qquad \text{if } \theta = \pm\pi/3, \tag{3.4a}$$

$$C_{34} = C_{14} = C_{24} = C_{56} = 0 \quad \text{if } \theta \neq \pm\pi/3. \tag{3.4b}$$

The remaining four equations in (3.2) are trivial identities when $\theta = \pi/2$. If $\theta \neq \pi/2$, the seventh, sixth, fifth and eighth equations of (3.2), in that order, yield

$$C_{44} = C_{55}, \quad C_{11} = C_{22}, \quad C_{13} = C_{23}, \tag{3.5a}$$

$$(C_{11} - C_{12} - 2C_{66})\cos 2\theta = 0. \tag{3.5b}$$

It is readily shown from (3.4a,b) and (3.5a,b) that the material is *orthotropic* when $\theta = \pi/2$ for which (Voigt, 1910; Love, 1927; Cowin and Mehrabadi, 1987, 1995)

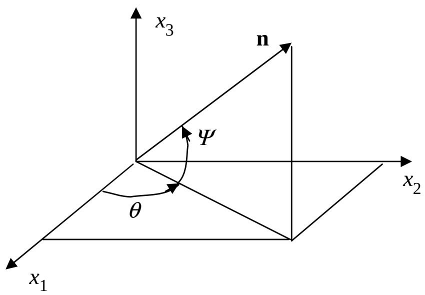

Fig. 1. The normal **n** to a symmetry plane.

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}, \tag{3.6}$$

and *tetragonal* when $\theta = \pm\pi/4$ for which

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{66} \end{bmatrix}. \tag{3.7}$$

It is *trigonal* when $\theta = \pm\pi/3$ for which

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & C_{14} \\ & & & & & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix}, \tag{3.8}$$

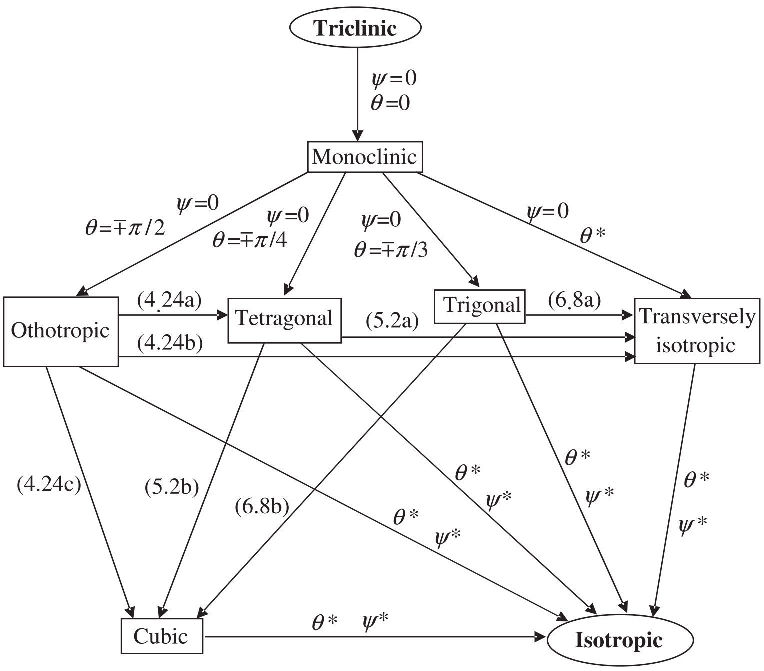

Fig. 2. Reduction of eight elastic symmetries by a successive imposition of a symmetry plane (see Fig. 1 in Chadwick et al., 2001). The number in the parentheses refers to the equation in the paper. The θ^*, ψ^* imply any θ, ψ not used in the elastic symmetry.

and *transversely isotropic* when $\theta \neq \pm\pi/2$, $\pm\pi/3$ or $\pm\pi/4$ for which

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix}. \tag{3.9}$$

These results are shown in Fig. 2 (see Fig. 1 in Chadwick et al., 2001). In (3.7)–(3.9) the x_3-axis can be identified as the *principal axis* of the material. Eqs. (3.7)–(3.9) can be modified for the cases when the x_1- or x_2-axis is the principal axis.

The number of independent elastic constants is 21 for a triclinic material and 13 for a monoclinic material shown in (2.8). The number of independent elastic constants for orthotropic, tetragonal, trigonal, transversely isotropic shown in (3.6)–(3.9) are nine, six, six and five, respectively. Some of the elastic constants can be made to vanish by choosing a new coordinate system. Hence the number of elastic constants can be reduced (Cowin and Mehrabadi, 1995).

In the following sections we add a third symmetry plane to the materials shown in (3.6)–(3.9). The matrices **U** and **V** reduce to diagonal matrices for these materials.

4. Adding a symmetry plane to an orthotropic material

In this section we add a new symmetry plane to an orthotropic material. Let the normal **n** to the new symmetry plane be given by (Fig. 1)

$$\mathbf{n}^{\mathrm{T}} = [\cos\psi\cos\theta \quad \cos\psi\sin\theta \quad \sin\psi], \tag{4.1}$$

$$-\pi/2 < \theta \leqslant \pi/2, \quad -\pi/2 < \psi \leqslant \pi/2. \tag{4.2}$$

The **n** for θ and ψ not covered in (4.2) is the negative of **n** given in (4.1). The elastic matrix **C** for an orthotropic material is shown in (3.6). Although (3.6) is obtained by imposing two symmetry planes whose normals are $(\theta, \psi) = (0, 0)$ and $(\pi/2, 0)$, it automatically has a symmetry plane with the normal $\psi = \pi/2$. Hence these (θ, ψ) are excluded in this section.

We employ (3.2) and choose

$$\mathbf{m}_1^{\mathrm{T}} = [\sin\theta \quad -\cos\theta \quad 0], \quad \mathbf{m}_2^{\mathrm{T}} = [0 \quad -\sin\psi \quad \cos\psi\sin\theta]. \tag{4.3}$$

The first four equations of (3.2) are trivial identities when $\theta = 0$ or $\pi/2$ while the last four equations are trivial identities when $\psi = 0$ or $\theta = 0$. We will study these special cases first.

(i) When $\psi = 0$, the first four equations of (3.2) give

$$C_{11} - C_{22} = C_{23} - C_{13} = C_{44} - C_{55} = (C_{11} - C_{12} - 2C_{66})(1 - \tan^2\theta) = 0. \tag{4.4}$$

Hence the material is tetragonal shown in (3.7) when $\theta = \pm\pi/4$ and transversely isotropic shown in (3.9) when $\theta \neq \pm\pi/4$.

(ii) When $\theta = \pi/2$, the last four equations of (3.2) give

$$C_{22} - C_{33} = C_{13} - C_{12} = C_{55} - C_{66} = (C_{22} - C_{23} - 2C_{44})\cos 2\psi = 0. \tag{4.5}$$

Hence the material is tetragonal when $\psi = \pm\pi/4$ and transversely isotropic when $\psi \neq \pm\pi/4$. The x_1-axis is the principal axis for the tetragonal and transversely isotropic materials.

(iii) When $\theta = 0$, all eight equations of (3.2) are trivial identities. This is so because the vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ in (4.3) are co-directional, and are not independent vectors. For $\theta = 0$, we choose

$$\mathbf{m}_1^{\mathrm{T}} = [0 \quad 1 \quad 0], \quad \mathbf{m}_2^{\mathrm{T}} = [-\sin\psi \quad 0 \quad \cos\psi\cos\theta]. \tag{4.6}$$

It can then be shown that the first four equations of (3.2) are trivial identities while the last four equations give

$$C_{11} - C_{33} = C_{23} - C_{12} = C_{44} - C_{66} = (C_{11} - C_{13} - 2C_{55})\cos 2\psi = 0. \tag{4.7}$$

The material is tetragonal when $\psi = \pm\pi/4$ and transversely isotropic when $\psi \neq \pm\pi/4$. The x_2-axis is the principal axis for the tetragonal and transversely isotropic materials.

We next consider the general case $\psi \neq 0$ and $\theta \neq 0$ or $\pi/2$. The first three equations of (3.2) give

$$C_{11} - C_{22} = C_{23} - C_{13} = C_{44} - C_{55} = (C_{11} - C_{12} - 2C_{66})(1 - \tan^2\theta). \tag{4.8}$$

The fifth, sixth and seventh equations yield

$$C_{22} - C_{33} = C_{13} - C_{12} = C_{55} - C_{66} = (C_{11} - C_{12} - 2C_{66})(3\sin^2\theta - 1). \tag{4.9}$$

Use of (4.8) and (4.9) in the fourth and eighth equations of (3.2) leads to

$$(C_{11} - C_{12} - 2C_{66})(1 - 2\cos^2\theta)(1 - 4\sin^2\psi) = 0, \tag{4.10}$$

$$(C_{11} - C_{12} - 2C_{66})[(4\sin^2\psi - 1) - 8(1 - 3\sin^2\theta)(1 - 3\cos^2\psi)\cos^2\psi] = 0. \tag{4.11}$$

There are several possibilities for (4.8)–(4.11) to hold.

(iv) When $\theta = \pm\pi/4$, (4.10) holds while (4.8) and (4.11) reduce to

$$C_{11} - C_{22} = C_{23} - C_{13} = C_{44} - C_{55} = 0, \tag{4.12}$$

$$(C_{11} - C_{12} - 2C_{66})(1 - 2\cos^2\psi) = 0. \tag{4.13}$$

If $\psi = \pm\pi/4$, (4.13) holds and (4.9) simplifies to

$$C_{22} - C_{33} = C_{13} - C_{12} = C_{55} - C_{66} = \tfrac{1}{2}(C_{11} - C_{12} - 2C_{66}). \tag{4.14}$$

The elastic matrix $\mathbf{C}$ that satisfies (4.12) and (4.14) has the structure

$$\mathbf{C} = \begin{bmatrix} \kappa-\alpha & \lambda+\alpha & \lambda & 0 & 0 & 0 \\ & \kappa-\alpha & \lambda & 0 & 0 & 0 \\ & & \kappa & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu+\alpha \end{bmatrix}, \tag{4.15a}$$

where

$$\alpha = \tfrac{1}{2}(\kappa - \lambda) - \mu. \tag{4.15b}$$

There are only three independent elastic constants κ, λ and μ. The elements of $\mathbf{C}$ have the relations

$$C_{44} = \tfrac{1}{2}(C_{11} - C_{12}), \quad C_{66} = \tfrac{1}{2}(C_{33} - C_{13}) = \tfrac{1}{2}(C_{11} + C_{12}). \tag{4.16}$$

When the coordinate system is rotated about the x_3-axis an angle $\pi/4$, the matrix $\mathbf{C}$ in (4.15a) referred to the rotated coordinate system can be shown to be

$$\mathbf{C} = \begin{bmatrix} \kappa & \lambda & \lambda & 0 & 0 & 0 \\ & \kappa & \lambda & 0 & 0 & 0 \\ & & \kappa & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}. \tag{4.17}$$

This represents a *cubic* material. Hence the $\mathbf{C}$ in (4.15a) with the α given in (4.15b) represents a cubic material.

If $\psi \neq \pm\pi/4$, (4.13) gives

$$C_{11} - C_{12} - 2C_{66} = 0. \tag{4.18}$$

Together with (4.12) and (4.14) we have

$$\mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}. \tag{4.19}$$

This represents an *isotropic* material in which λ and μ are the Lame constants.

(v) When $\psi = \pm\pi/6$ and $\theta = \pm\sin^{-1}(1/\sqrt{3})$, (4.10) and (4.11) hold while (4.8) and (4.9) reduce to

$$\begin{aligned} &C_{11} - C_{22} = C_{23} - C_{13} = C_{44} - C_{55} = \tfrac{1}{2}(C_{11} - C_{12} - 2C_{66}), \\ &C_{22} - C_{33} = C_{13} - C_{12} = C_{55} - C_{66} = 0. \end{aligned} \tag{4.20}$$

The elastic matrix $\mathbf{C}$ that satisfies (4.20) has the structure

$$\mathbf{C} = \begin{bmatrix} \kappa & \lambda & \lambda & 0 & 0 & 0 \\ & \kappa - \alpha & \lambda + \alpha & 0 & 0 & 0 \\ & & \kappa - \alpha & 0 & 0 & 0 \\ & & & \mu + \alpha & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}, \tag{4.21}$$

where α is defined in (4.15b). The $\mathbf{C}$ in (4.21) becomes the $\mathbf{C}$ in (4.17) when the coordinate system is rotated about the x_1-axis an angle $\pi/4$. Hence (4.21) represents a cubic material.

(vi) When $\psi = \pm\pi/6$ and $\theta = \pm\cos^{-1}(1/\sqrt{3})$, (4.10) and (4.11) hold while (4.8) and (4.9) reduce to

$$\begin{aligned} &C_{11} - C_{22} = C_{23} - C_{13} = C_{44} - C_{55} = -(C_{11} - C_{12} - 2C_{66}), \\ &C_{22} - C_{33} = C_{13} - C_{12} = C_{55} - C_{66} = -(C_{11} - C_{12} - 2C_{66}). \end{aligned} \tag{4.22}$$

The elastic matrix $\mathbf{C}$ that satisfies (4.22) has the structure

$$\mathbf{C} = \begin{bmatrix} \kappa - \alpha & \lambda & \lambda + \alpha & 0 & 0 & 0 \\ & \kappa & \lambda & 0 & 0 & 0 \\ & & \kappa - \alpha & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu + \alpha & 0 \\ & & & & & \mu \end{bmatrix}, \tag{4.23}$$

where α is defined in (4.15b). The **C** in (4.23) becomes the **C** in (4.17) when the coordinate system is rotated about the x_2-axis an angle $\pi/4$. Hence (4.23) represents a cubic material.

(vii) For all other θ and ψ, (4.10) and (4.11) give (4.18). Use of (4.18) in (4.8) and (4.9) leads to (4.19) so that the material is isotropic.

In summary, when a new symmetry plane is added to an orthotropic material shown in (3.6), the material is tetragonal when

$$(\theta, \psi) = (\pm\pi/4, 0),\ (0, \pm\pi/4),\ (\pi/2, \pm\pi/4), \tag{4.24a}$$

transversely isotropic when

$$\begin{aligned} &\psi = 0, \quad \theta \neq 0, \pm\pi/4, \pi/2, \\ &\theta = 0, \pi/2, \quad \psi \neq 0, \pm\pi/4, \pi/2, \end{aligned} \tag{4.24b}$$

cubic when

$$(\theta, \psi) = (\pm\phi_1, \pi/6),\ (\pm\phi_2, \pi/6),\ (\pm\pi/4, \pi/4),\ (\pm\phi_1, -\pi/6),\ (\pm\phi_2, -\pi/6),\ (\pm\pi/4, -\pi/4), \tag{4.24c}$$

and isotropic for any other (θ, ψ). In (4.24c),

$$0 < \phi_1 = \sin^{-1}(1/\sqrt{3}) < \pi/2, \quad 0 < \phi_2 = \cos^{-1}(1/\sqrt{3}) < \pi/2. \tag{4.25}$$

5. Adding a symmetry plane to a tetragonal material

A tetragonal material of (3.7) is a special orthotropic material of (3.6) when

$$C_{11} = C_{22}, \quad C_{13} = C_{23}, \quad C_{44} = C_{55}. \tag{5.1}$$

Therefore the derivation is simpler than the one for an orthotropic material presented in the previous section. We list below only the results of the derivation.

(i) When $\psi = 0$ the material is transversely isotropic with the x_1-axis being the principal axis.
(ii) When $\theta = 0$ or $\pi/2$, the material is cubic if $\psi = \pm\psi/4$ and isotropic if $\psi \neq \pm\pi/4$. The elastic matrix **C** for the cubic material is given in (4.17).
(iii) For $\theta \neq 0$ or $\pi/2$ and $\psi \neq 0$ or $\pi/2$, the material is cubic when $\theta = \pm\pi/4$ and $\psi = \pm\pi/4$, and isotropic otherwise. The elastic matrix **C** for the cubic material is given in (4.15).

In summary, when a new symmetry plane is added to a tetragonal material shown in (3.7), the material is transversely isotropic if

$$\psi = 0, \quad \theta \neq 0, \pm\pi/4, \pi/2, \tag{5.2a}$$

cubic if

$$\begin{aligned} &\psi = \pi/4, \quad \theta = 0, \pm\pi/4, \pi/2, \\ &\psi = -\pi/4, \quad \theta = 0, \pm\pi/4, \pi/2, \end{aligned} \tag{5.2b}$$

and isotropic for any other (θ, ψ).

6. Adding a symmetry plane to a trigonal material

The elastic matrix **C** for a trigonal material is given in (3.8). If the coordinate system is rotated about the x_3-axis an angle $\pi/6$, the three C_{14} in (3.8) change sign (Ting, 2000). We employ (3.2) in which **n**, $\mathbf{m}_1$, $\mathbf{m}_2$ are shown in (4.1) and (4.3). The first two equations of (3.2) are trivial identities. As before the special cases of $\psi = 0$ and $\theta = 0, \pi/2$ are studied first.

(i) When $\psi = 0$, all equations in (3.2) are trivial identities except the seventh, which simplifies to

$$C_{14}(4\cos^2\theta - 1)\sin^2\theta = 0. \tag{6.1}$$

Since $\theta = 0$ and $\pm\pi/3$ are the normals to the symmetry plane of a trigonal material, they are excluded here. Hence $C_{14} = 0$, and the material is transversely isotropic.

(ii) When $\psi = \pi/2$, the fifth and sixth equations of (3.2) are trivial identities. The eighth equation gives $C_{14} = 0$ which satisfies the rest of the equations in (3.2). Hence the material is transversely isotropic.

(iii) For $\theta = \pi/2$, it can be shown that the material is isotropic unless $\psi = \pm\phi_1$ or $\pm\phi_2$ where ϕ_1, ϕ_2 are defined in (4.25). If $\psi = \pm\phi_1$, we have

$$C_{33} - C_{11} = C_{12} - C_{13} = C_{66} - C_{44} = \pm C_{14}/\sqrt{2}. \tag{6.2}$$

If $\psi = \mp\phi_2$, we also have (6.2). The elastic matrix **C** that satisfies (6.2) has the structure

$$\mathbf{C} = \begin{bmatrix} \kappa - 3\gamma & \lambda + \gamma & \lambda + 2\gamma & \mp\sqrt{2}\gamma & 0 & 0 \\ & \kappa - 3\gamma & \lambda + 2\gamma & \pm\sqrt{2}\gamma & 0 & 0 \\ & & \kappa - 4\gamma & 0 & 0 & 0 \\ & & & \mu + 2\gamma & 0 & 0 \\ & & & & \mu + 2\gamma & \mp\sqrt{2}\gamma \\ & & & & & \mu + \gamma \end{bmatrix}, \tag{6.3a}$$

where

$$\gamma = \tfrac{1}{6}(\kappa - \lambda - 2\mu). \tag{6.3b}$$

It has only three independent elastic constants, and satisfies the relation

$$C_{66} = \tfrac{1}{2}(C_{11} - C_{12}). \tag{6.4}$$

The cubic material defined in (4.17) has six symmetry planes. The normals to the symmetry planes are along the coordinate axes and on the coordinate planes making an angle $\pi/4$ with the coordinate axes. The normals on the coordinate planes make an angle $\pi/3$ to each other (Chadwick et al., 2001). An octahedral plane that is equally inclined to the coordinate axes contains three such normals. Let x_i^* $(i = 1, 2, 3)$ be a new coordinate system in which the x_1^*- and x_2^*-axes are on an octahedral plane with the x_1^*-axis being one of the normals. One choice of such coordinate system is

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{6.5}$$

The elastic matrix **C** of (4.17), when referred to the x_i^* coordinate system, takes the form (6.3a,b) with the lower sings for C_{14}, C_{24} and C_{56}. If we chose a different octahedral plane or a different normal on the octahedral plane as the x_1^*-axis, the upper signs in (6.3a) apply. Hence the **C** in (6.3a,b) represents a cubic material.

(iv) When $\theta = 0$, the $\mathbf{m}_1$ and $\mathbf{m}_2$ in (4.3) are co-directional, and hence are not independent vectors. We replace $\mathbf{m}_1$ and $\mathbf{m}_2$ by the ones given in (4.6). It can then be shown that the material is isotropic for all $\psi \neq 0$ or $\pi/2$.

(v) For other θ and ψ for which $\theta \neq 0$ or $\pi/2$ and $\psi \neq 0$ or $\pi/2$, the third equation of (3.2) gives

$$C_{14}(1 - 4\sin^2\theta) = 0. \tag{6.6}$$

The fourth and eighth equations of (3.2) are compatible if

$$C_{14}(1 - 3\sin^2\psi)(1 - 3\cos^2\psi) = 0. \tag{6.7}$$

It can be shown that, when $\theta = \pi/3$ and $\psi = \mp\phi_1$ or $\pm\phi_2$, we have (6.2). The ϕ_1 and ϕ_2 are defined in (4.25). If $\theta = -\pi/3$ and $\psi = \pm\phi_1$ or $\mp\phi_2$, we also have (6.2). Thus the material is cubic. Otherwise it is isotropic.

In summary, when a new symmetry plane is added to a trigonal material of (3.8), the material is transversely isotropic if

$$\psi = 0, \quad \theta \neq 0, \pm\pi/3 \quad \text{or} \quad \psi = \pi/2, \tag{6.8a}$$

cubic if

$$\psi = \pm\phi_1, \pm\phi_2 \quad \text{and} \quad \theta = 0, \pm\pi/6, \pi/2, \tag{6.8b}$$

and isotropic for any other (θ, ψ).

7. Transversely isotropic and cubic materials

It is very simple to show that when a new symmetry plane is added to a transversely isotropic or cubic material, the material is isotropic. Since any plane is a symmetry plane for an isotropic material, no further reduction is possible. The results obtained in Sections 3–7 are summarized in Fig. 2.

The structure of the elastic matrix **C** for a cubic material in (4.17) is the familiar one in the literature. Less familiar are the **C** shown in (4.15a,b) and (6.3a,b). The **C** in (4.15a,b) resembles a tetragonal material. As shown in Section 5, when a new symmetry plane with the normal $\theta = \pm\pi/4$ and $\psi = \pm\pi/4$ is added to a tetragonal material of (3.7), we obtain the cubic material shown in (4.15a,b). The **C** in (6.3a,b) resembles a trigonal material in view of (6.4). The normals to the symmetry planes of a trigonal material make an angle $\pi/3$. The normals to the symmetry planes of a cubic material given in (4.17) are along the coordinate axes and on the coordinate planes making an angle $\pi/4$ with the coordinate axes. The normals that lie on the coordinate planes make an angle $\pi/3$ with each other. It is therefore not surprising that a cubic material resembles a trigonal material. This fact has been pointed out by Chadwick et al. (2001).

8. Concluding remarks

By imposing one symmetry plane at a time we have proved that there are exactly eight elastic symmetries for a linear anisotropic elastic material. The results are summarized in Fig. 2. Fig. 2 is re-drawn from Fig. 1 in Chadwick et al. (2001) but we have added explicitly what symmetry planes are needed in reducing to the next elastic symmetry. It is seen from Fig. 2 that the shortest routes from triclinic, which has no symmetry plane, to isotropic is triclinic–monoclinic–either orthotropic, tetragonal, trigonal or transversely isotropic–isotropic. These four routes require imposition of three symmetry planes. The longest route is triclinic–monoclinic–orthotropic–tetragonal–cubic–isotropic. This route requires imposition of five symmetry planes. The routes that require imposition of four symmetry planes are triclinic–monoclinic, and then

(i) orthotropic–tetragonal–isotropic, (ii) orthotropic–transversely isotropic–isotropic, (iii) orthotropic–cubic–isotropic, (iv) tetragonal–transversely isotropic–isotropic, (v) tetragonal–cubic–isotropic, (vi) trigonal–transversely isotropic–isotropic, or (vii) trigonal–cubic–isotropic. Thus it takes as few as three, and at most five, symmetry planes to reduce a triclinic material to an isotropic material.

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